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Spacetime Structure on Quantum Lattice

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1. INTRODUCTION

The message of general relativity is the following: there is no "physical field" called the gravitational field. Gravitation is a manifestation of certain properties of spacetime. Therefore, quantization of gravity would mean to "quantize" the spacetime structure itself.

Instead of the usual field-theoretic approach of quantization of general relativity, I suggest to develop a theory of spacetime which is intrinsically based on quantum theory.

The primary object of the current formulation of general relativity is an "underlying set" on which the (causal, topological, geometrical) structure of spacetime is defined. The elements of this set are called "events." The meaning of an event is rather vague, especially if we want to abstain from tautology: defining an event with reference to the spacetime structure itself.

In quantum theory an event means a possible result of a possible measurement or observation performed on a physical object. I consider the set of all physical events belonging to the whole universe, rather than to a separate physical system.

This set of events has an immanent "logical" structure, a non-Boolean lattice structure on which the quantum mechanical probability theory can be defined. The (quantum) spacetime is a structure defined over the quantum lattice of physical events.

The elements of the quantum lattice play the same role as the subsets of the (classical) spacetime. In general, a physical event corresponds to a subset of spacetime. A subset of spacetime should be interpreted as a complex of events contained in the subset. For instance, the event "cloud camera D has detected particle γ" is the conjunction of events $p_1, p_2, p_3, \ldots$ (see Figure 1). Therefore, the *union* of subsets corresponds to the *conjunction* of the corrersponding physical events, while the *common part* of subsets means

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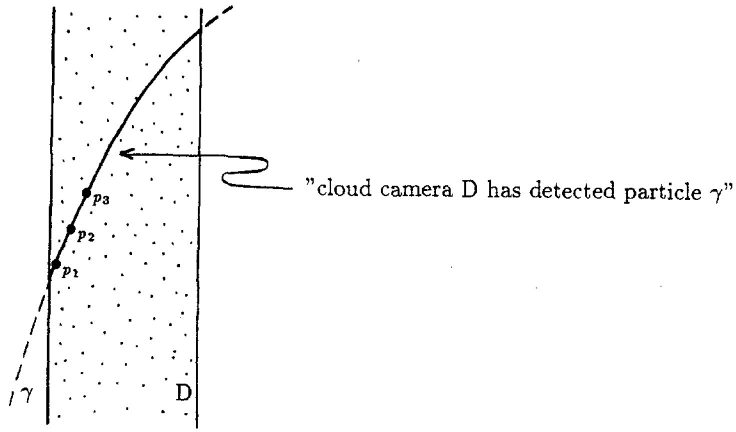


Fig. 1. There are physical events which correspond to subsets of spacetime.

the *disjunction*. That means in classical physics there is a dual isomorphism between the lattice of physical events and the subset lattice of spacetime. This correspondence cannot be correct in quantum theory, because the lattice of physical events is not Boolean.

A possible resolution of this inadequacy is if the Boolean subset lattice is exchanged for the dual of the quantum lattice, and if the whole spacetime structure is built on this ground up.

2. QUANTUM CAUSAL STRUCTURE

As an initial step we are going to "quantize" the *causal structure* of spacetime. Recall the Kronheimer and Penrose (1967) axioms of a causal space. Let X be an underlying set. The causal relation $\prec$ and the chronological relation $\ll$ satisfies the following axioms ($x, y, z \in X$):

K1. $x \prec x$
K2. If $x \prec y$ and $y \prec z$, then $x \prec z$
K3. From $x \prec y$ and $y \prec x$ it follows that $x = y$
K4. Not $x \ll x$
K5. If $x \ll y$, then $x \prec y$
K6. If $x \prec y$ and $y \ll z$, then $x \ll z$
K7. If $x \ll y$ and $y \prec z$, then $x \ll z$

The *causal future* (*past*) set and the *chronological future* (*past*) set of a set A are defined as follows:

$$J^{+(-)}(A) := \{x \in X \mid \text{there exists } a \in A \text{ such that } a \prec x (x \prec a)\}$$

$$I^{+(-)}(A) := \{x \in X \mid \text{there exists } a \in A \text{ such that } a \ll x (x \ll a)\}$$

One can express axioms by the subsets of X:

P1. $A \subset J^{\pm}(A)$
P2. $I^{\pm}(A) \subset J^{\pm}(A)$
P3. $J^{\pm}(J^{\pm}(A)) = J^{\pm}(A)$
P4. $J^{\pm}(A \cup B) = J^{\pm}(A) \cup J^{\pm}(B)$
P5. $I^{\pm}(A \cup B) = I^{\pm}(A) \cup I^{\pm}(B)$
P6. $J^{\pm}(I^{\pm}(A)) = I^{\pm}(J^{\pm}(A)) = I^{\pm}(A)$
P7. $I^{\pm}(\mathcal{M}) \neq \varnothing$
P8. $\{x\} \not\subset I^{\pm}(\{x\})$
P9. $\{x\} \subset J^{+}(\{y\}) \Leftrightarrow \{y\} \subset J^{-}(\{x\})$
P10. $\{x\} \subset I^{+}(\{y\}) \Leftrightarrow \{y\} \subset I^{-}(\{x\})$
P11. $J^{\pm}(\{x\}) = J^{\pm}(\{y\}) \Rightarrow x = y$

Here x and y are atoms of S; $A, B \in S$; and $\mathcal{M}$ and $\varnothing$ denote the maximal element and the minimal element of S.

It is obvious that two pairs of maps

$$J^{\pm}: \quad P(X) \to P(X)$$

$$I^{\pm}: \quad P(X) \to P(X)$$

satisfying P1–P11 define a causal structure on X via the relations

$$x \prec y \Leftrightarrow y \subset J^{+}(x)$$

$$x \ll y \Leftrightarrow y \subset I^{+}(x)$$

Replace now the subset lattice $(P(X), \cap, \cup)$ by the dual of a quantum lattice of physical events $(S, \wedge, \vee)$. A *quantum causal structure* (Szabó, 1986) is given by the causal and chronological future and past defined as maps

$$J^{\pm}: \quad S \to S$$

$$I^{\pm}: \quad S \to S$$

with the following properties:

Q1. $A < J^{\pm}(A)$
Q2. $I^{\pm}(A) < J^{\pm}(A)$
Q3. $J^{\pm}(J^{\pm}(A)) = J^{\pm}(A)$
Q4. $J^{\pm}(A \vee B) = J^{\pm}(A) \vee J^{\pm}(B)$
Q5. $I^{\pm}(A \vee B) = I^{\pm}(A) \vee I^{\pm}(B)$
Q6. $J^{\pm}(I^{\pm}(A)) = I^{\pm}(J^{\pm}(A)) = I^{\pm}(A)$
Q7. $I^{\pm}(\mathcal{M}) \neq \varnothing$
Q8. $x \not< I^{\pm}(x)$
Q9. $x < J^{+}(y) \Leftrightarrow y < J^{-}(x)$
Q10. $x < I^{+}(y) \Leftrightarrow y < I^{-}(x)$
Q11. $J^{\pm}(x) = J^{\pm}(y) \Rightarrow x = y$

Here x and y are atoms of S; $A, B \in S$; and $\mathcal{M}$ and $\varnothing$ denote the maximal element and the minimal element of S.

The causality and chronology relations can be defined as

$$A \prec B \Leftrightarrow B < J^{\pm}(A) \quad \text{or} \quad A < J^{\pm}(B)$$

$$A \ll B \Leftrightarrow B < I^{\pm}(A) \quad \text{or} \quad A < I^{\pm}(B)$$

A and B are called *spatially separated* if neither $A \prec B$ nor $B \prec A$ holds.

If S is a Boolean lattice, it can be represented by a suitable subset lattice and the quantum causality leads to the usual causality on an "underlying set" of Kronheimer and Penrose.

3. ALEXANDROV T-STRUCTURE AND DIMENSION

Causality is the most important part of the structure of spacetime. On the ground of the causal structure one can introduce further structures (Szabó, 1989*a*). In the absence of an underlying set one cannot introduce a topology which could make the quantum spacetime a topological space. Fortunately, one can generalize the notion of topology: a so called T-structure is defined via the closure operation:

$$\bar{}: \quad S \to S, \quad A \mapsto \bar{A}$$

1. $A < \bar{A}$
2. $\overline{(A \vee B)} = \bar{A} \vee \bar{B}$
3. $\bar{\varnothing} = \varnothing$
4. $\bar{\bar{A}} = \bar{A}$

An element A is *open* iff the orthocomplement $A^{\perp}$ is *closed*, that is,

$$\overline{A^{\perp}} = A^{\perp}$$

As is well known, the causal structure on an underlying set singles out a topology called the Alexandrov topology which is the coarsest topology in which each chronological future and past is open. One can introduce the *Alexandrov T-structure* in a similar way: as the coarsest T-structure in which each $I^{\pm}(A)$ is open.

On the ground of the Alexandrov T-structure one can introduce the dimension of a "quantum spacetime" by analogy with the covering dimension of topological spaces. The *dimension* of a quantum causal structure at a given atom x is the smallest integer $d(x)$ such that for each open covering there is a refined open covering with overlap $d(x)+1$ at x.

4. AN EXAMPLE: EPR EXPERIMENT

Let us consider a spin-0 system which consists of two spin-1/2 particles. The spin part of the state vector is given by

$$\psi = \frac{1}{\sqrt{2}} (\mathbf{u}_{\mathbf{n}}^{+}(1) \otimes \mathbf{u}_{\mathbf{n}}^{-}(2) - \mathbf{u}_{\mathbf{n}}^{-}(1) \otimes \mathbf{u}_{\mathbf{n}}^{+}(2)) \tag{1}$$

where $\mathbf{u}_{\mathbf{n}}^{\pm}(1)$ describes a state in which particle 1 has spin up or down, respectively, along the direction $\mathbf{n}$, and $\mathbf{u}_{\mathbf{n}}^{\pm}(2)$ has a similar meaning concerning particle 2. We now assume that the two particles are isolated from each other, by separating them spatially.

According to the *classical* spacetime causality, any observation carried out on one of the particles cannot have any effect on the other one. Suppose we measure the spin of particle 1 along the direction **a**. The outcome is not predetermined by the state vector ψ. But it determines entirely the outcome of measurement carried out on particle 2. This means we find a *manifest correlation* between the physical events corresponding to subspaces generated by vectors $\mathbf{u}_\mathbf{a}^+(1)$ and $\mathbf{u}_\mathbf{a}^-(2)$ or $\mathbf{u}_\mathbf{a}^-(1)$ and $\mathbf{u}_\mathbf{a}^+(2)$ in spatially separated regions. Finding correlation between two events A and B, one can imagine two different possibilities. The first is that it is a direct type of correlation, that is, one of the events has a direct influence on the other. The second possibility is that it is a common-cause type correlation, which means there is a third event C which has direct influence on both A and B (see Figure 2).

The first possibility contradicts the classical structure of spacetime. In the second case the probabilities should satisfy the Bell inequalities, while those which are measured (and calculated from the quantum mechanics) in the spin correlation experiments violate the Bell inequalities. Therefore, the second case is also excluded.

Consider a model quantum lattice which is a finite sublattice of the subspace lattice of $H^2 \otimes H^2$ generated by the following one-dimensional subspaces as atoms:

$$A = \mathbf{u}_\mathbf{a}^+(1) \otimes \mathbf{u}_\mathbf{a}^-(2)$$

$$B = \mathbf{u}_\mathbf{a}^-(1) \otimes \mathbf{u}_\mathbf{a}^+(2)$$

$$C = \mathbf{u}_\mathbf{a}^+(1) \otimes \mathbf{u}_\mathbf{a}^+(2)$$

$$D = \mathbf{u}_\mathbf{a}^-(1) \otimes \mathbf{u}_\mathbf{a}^-(2)$$

$$E = \mathbf{u}_\mathbf{a}^+(1) \otimes \mathbf{u}_\mathbf{a}^-(2) + \mathbf{u}_\mathbf{a}^-(1) \otimes \mathbf{u}_\mathbf{a}^+(2)$$

$$F = \mathbf{u}_\mathbf{a}^+(1) \otimes \mathbf{u}_\mathbf{a}^-(2) - \mathbf{u}_\mathbf{a}^-(1) \otimes \mathbf{u}_\mathbf{a}^+(2)$$

The lattice generated by them is shown in Figure 3.

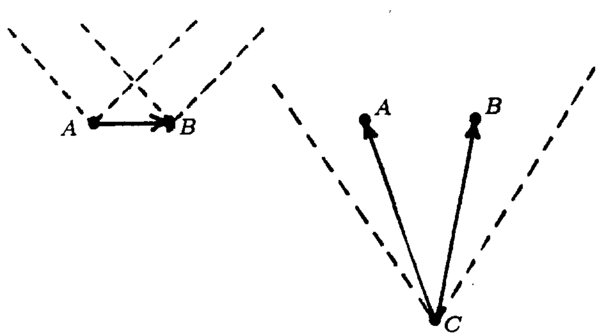


Fig. 2. The direct and the common-cause types of correlations.

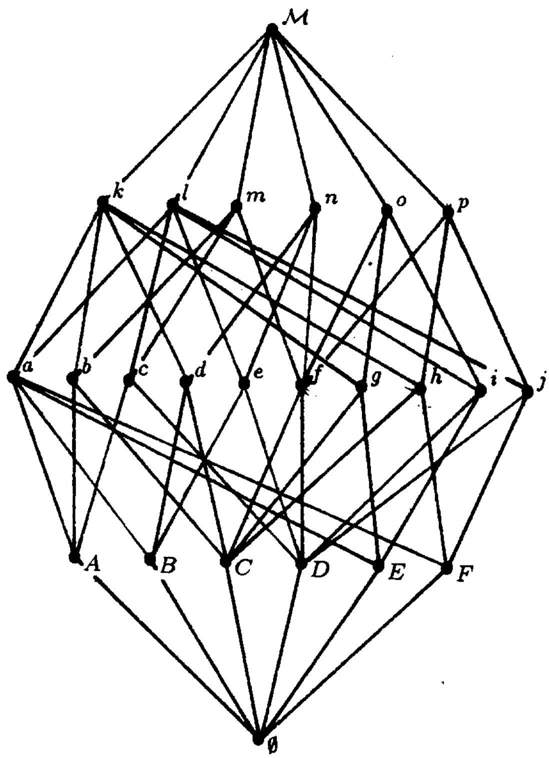


Fig. 3. Model quantum lattice describing the EPR events.

The quantum causal structure is based on the dual of the quantum lattice of events. Denote by S the dual of the model quantum lattice in question (see Figure 4).

The crucial EPR events correspond to the following elements of these lattices:

$b = 15 =$ particle 1 has spin "up" along direction **a**

$e = 12 =$ particle 1 has spin "down" along direction **a**

$d = 13 =$ particle 2 has spin "up" along direction **a**

$c = 14 =$ particle 2 has spin "down" along direction **a**

Of course, there can exist a number of quantum causal structures of S satisfying the axioms Q1–Q11, analogously to conventional relativity, where on a given manifold one can also introduce many different metrics satisfying the conditions required for a spacetime.

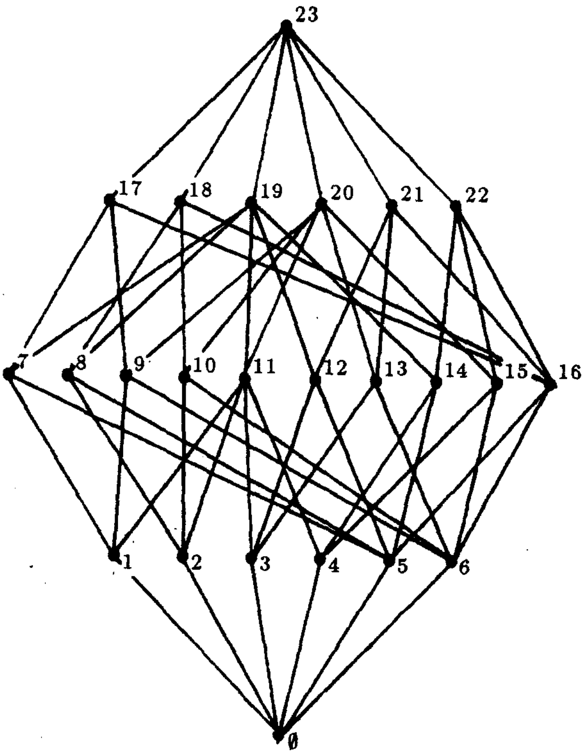
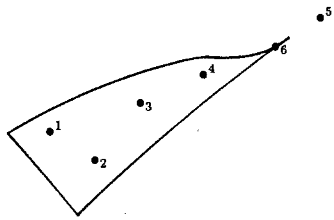


Fig. 4. The dual of the model quantum lattice.

I have examined by computer all possible quantum causal structures on this lattice, i.e., all quadruples $(J^{\pm}, I^{\pm})$ of maps from S to itself (from the 96^{24} ones) which satisfy the axioms. There exist 35 different quantum causal structures of lattice S. One of them is shown in Table I. Considering these causal structures, one finds the following surprising result. In each quantum causal structure the events 12 and 13 as well as 15 and 14 *are not spatially separated.* Consequently, having assumed that the intrinsic causality of physical events is described by a quantum causal structure, we have found that

Table I. One of the 35 Quantum Causal Structures

A	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$A^{\perp}$	18	17	22	21	20	19	10	9	8	7	16	15	14	13	12	11	2	1	6	5	4	3	0
$J^{+}(A)$	9	10	13	15	5	6	17	18	9	10	20	21	13	22	15	16	17	18	23	20	21	22	23
$I^{+}(A)$	6	6	6	6	0	0	6	6	6	6	6	6	6	6	6	0	6	6	6	6	6	6	6
$J^{-}(A)$	1	2	3	4	5	20	7	8	20	20	11	12	20	14	20	23	23	23	19	20	23	23	23
$I^{-}(A)$	0	0	0	0	0	11	0	0	11	11	0	0	11	0	11	11	11	11	0	11	11	11	11
A	19	19	19	19	5	16	19	19	23	23	19	19	23	19	23	16	23	23	19	23	23	23	23



x	1	2	3	4	5	6
d(x)	2	2	2	2	0	1

Fig. 5. The dimensions of the model quantum spacetime.

the EPR paradox is resolved in the sense that causality does not exclude the possibility of the direct-type correlations (Szabó, 1989*b*).

The closure operation of the Alexandrov T-structure belonging to the considered quantum causal structure is also shown in Table I. According to this T-structure the closed elements are 0, 5, 16, 19, and 23, while the open elements are 23, 20, 11, 6, and 0. The Alexandrov T-structure determines the dimension of this model quantum spacetime (see Figure 5).

We have seen an example for a quantum causal structure on a finite model quantum lattice. Finally, I mention that the problem of the existence of quantum causal structures on the whole infinite quantum lattice (i.e., on the subspace lattice of a Hilbert space) seems to be difficult. One can easily construct structures on the Hilbert lattice which satisfy Q1–Q7. Requirements Q8–Q11 play a double role. On one hand they provide a correspondence between futures and pasts, and on the other hand they require the same properties for the atoms of the lattice which characterize the points of a classical spacetime. The problem of Q8–Q11 has not been solved yet.

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